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Self-force in 2+1 electrodynamics

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Abstract

The radiation reaction problem for an electric charge moving in flat spacetime of three dimensions is discussed. The divergences stemming from the pointness of the particle are studied. A consistent regularization procedure is proposed, which exploits the Poincaré invariance of the theory. Effective equation of motion of radiating charge in an external electromagnetic field is obtained via the consideration of energy-momentum and angular momentum conservation. This equation includes the effect of the particle's own field. The radiation reaction is determined by the Lorentz force of point-like charge acting upon itself plus a non-local term which provides finiteness of the self-action.

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1. Introduction

There has been much recent interest [1, 2] in the renormalization procedure in classical electrodynamics of a point particle moving in flat spacetime of arbitrary dimensions. The main task is to derive the analogue of the well-known Lorentz–Dirac equation [3]. The Lorentz–Dirac equation is an equation of motion for a charged particle under the influence of an external force as well as its own electromagnetic field. (For a modern review see [4–6].)

A special attention in [1, 2] is devoted to the mass renormalization in 2+1 theory. (Note that electrodynamics in the Minkowski space $\mathbb{M}_3$ is quite different from the conventional 3+1 electrodynamics where one space dimension is reduced because of the symmetry of the specific problem. For example, small charged balls on a plane are interacted inversely with the square of the distance between them, while in $\mathbb{M}_3$ the Coulomb field of a small static charged disc scales as $|\mathbf{r}|^{-1}$.) An essential feature of 2+1 electrodynamics is that the Huygens principle does not hold and radiation develops a tail, as it is in curved spacetime of four dimensions [7] where electromagnetic waves propagate not just at the speed of light, but all speeds smaller than or equal to it.

In [1, 2] the self-force on a point-like particle is calculated from the local fields in the immediate vicinity of its trajectory. The schemes involve some prescriptions for subtracting away the infinite contributions to the force due to the singular nature of the field on the particle's world line. In [2] the procedure of regularization is based on the methods of functional analysis which are applied to the Taylor expansion of the retarded Green's function. The authors derive the covariant analogue of the Lorentz–Dirac equation which is something other than that obtained in [1]. Both the divergent self-energy absorbed by 'bare' mass of point-like charge, and the radiative term which leads an independent existence, are non-local. (They depend not only on the current state of motion of the particle, but also on its past history.)

In this paper, we develop a consistent regularization procedure which exploits the symmetry properties of 2+1 electrodynamics. It can be summarized as a simple rule which obeys the spirit of Dirac scheme of decomposition of vector potential of a point-like charge.

According to the scheme proposed by Dirac in his seminal paper [3], one can decompose the retarded Green's function associated with the four-dimensional Maxwell field equation $G^{\mathrm{ret}}(x,z) = G^{\mathrm{sym}}(x,z) + G^{\mathrm{rad}}(x,z)$. The first term, $G^{\mathrm{sym}}(x,z)$, is one-half sum of the retarded and the advanced Green's functions; it is just singular as $G^{\mathrm{ret}}(x,z)$. The second one, $G^{\mathrm{rad}}(x,z)$, is one-half of the retarded minus one-half of the advanced Green's functions; it satisfies the homogeneous wave equation. Convolving the source with Green's functions $G^{\mathrm{sym}}(x,z)$ and $G^{\mathrm{rad}}(x,z)$ yields the singular and the radiative parts of vector potential of a point-like charge, respectively.

The analogous decomposition of Green's function in curved spacetime is much more delicate because of richer causal structure. Detweiler and Whiting [9] modify the singular Green's function by means of a two-point function $v(x,z)$ which is symmetric in its arguments. It is constructed from the solutions of the homogeneous wave equation in such a way that a new symmetric Green's function $G^S(x,z) = G^{\mathrm{sym}}(x,z) + 1/(8\pi)v(x,z)$ has no support within the null cone.

The physically relevant solution to the wave equation is obviously the retarded solution. In [10], the Lorentz–Dirac equation is derived within the framework of retarded causality. Teitelboim substitutes the retarded Liénard–Wiechert fields in the electromagnetic field's stress–energy tensor. The author computes the flow of energy-momentum which flows across a tilted hyperplane which is orthogonal to particle's four-velocity at the instant of observation. The effective equation of motion is obtained in [10] via the consideration of energy-momentum conservation. Similarly, López and Villarroel [11] find out the total angular momentum carried by electromagnetic field of a point-like charge.

Outgoing waves carry energy-momentum and angular momentum; the radiation removes energy, momentum and angular momentum from the source which then undergoes a radiation reaction. It is shown [12] that the Lorentz–Dirac equation can be derived from the energy-momentum and angular momentum balance equations. In [13] the analogue of the Lorentz-Dirac equation in six dimensions is obtained via analysis of 21 conserved quantities which correspond to the symmetry of an isolated point particle coupled with electromagnetic field. (First, this equation was obtained by Kosyakov in [14] via the consideration of energy-momentum conservation. An alternative derivation was produced by Kazinski, Lyakhovich and Sharapov in [2].)

In non-local theories, the computation of Noether quantities is highly nontrivial. Quinn and Wald [15] study the energy-momentum conservation for point charge moving in curved spacetime. The Stokes' theorem is applied to the integral of flux of electromagnetic energy over the compact region $V(t^+,t^-)$. It is expanded to the limits $t^- \to -\infty$ and $t^+ \to +\infty$, so that finally the boundary of the integration domain involves smooth spacelike hypersurfaces at the remote past and in the distant future. The spacetime is asymptotically flat here. The authors prove that the net energy radiated to infinity is equal to the total work done on the particle by the electromagnetic self-force. (DeWitt–Brehme [7, 8] radiation reaction force is meant.) It is shown also [15] that the total work done by the gravitational self-force is equal to the energy radiated (in gravitational waves) by the particle. (The effective equation of motion of a point mass undergoing radiation reaction is obtained in [16]; see also review [17] where the motion of a point electric charge, a point scalar charge and a point mass in curved spacetime is considered in detail.)

In the present paper, we calculate the total flows of energy-momentum and angular momentum of the retarded field which flow across a hyperplane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$ associated with an unmoving observer. It is organized as follows. In section 2, we recall the retarded and the advanced Green's functions associated with the three-dimensional D'Alembert operator. Convolving them with the point source, we derive the retarded and the advanced vector potential and field strengths. In the appendix , we trace a series of stages in calculation of surface integral which gives the energy-momentum carried by the retarded electromagnetic field. We integrate the Maxwell energy-momentum tensor density over the variables which parametrize the surface of integration Σ_t. Resulting expression becomes a combination of the two-point functions depending on the state of particle's motion at instants t_1 and t_2 before the observation instant t. They are integrated over particle's world line twice. We arrange them in section 3. We split the momentum three-vector carried by electromagnetic field into singular and radiative parts by means of Dirac scheme which deals with the fields taken on the world line only. All diverging quantities have disappeared into the procedure of mass renormalization while radiative terms lead an independent existence. In analogous way we analyse the angular momentum of electromagnetic field. Total energy-momentum and total angular momentum of our particle plus field system depend on already renormalized particle's individual characteristics and *radiative* parts of Noether quantities. In section 4, we derive the effective equation of motion of radiating charge via analysis of balance equations. In section 5, we discuss the result and its implications.

2. Electromagnetic field in 2+1 theory

We consider an electromagnetic field produced by a particle with δ-shaped distribution of the electric charge e moving on a world line $\zeta \subset \mathbb{M}_3$ described by functions $z^\mu(\tau)$ of proper time τ. The Maxwell equations

$$F^{\alpha\beta}{}_{,\beta} = 2\pi j^\alpha, \tag{2.1}$$

where current density j^α is given by

$$j^\alpha = e \int_{-\infty}^{+\infty} \mathrm{d}\tau\, u^\alpha(\tau)\delta^{(3)}(y - z(\tau)), \tag{2.2}$$

governs the propagation of the electromagnetic field. $u^\alpha(\tau)$ denotes the (normalized) three-velocity vector $\mathrm{d}z^\alpha(\tau)/\mathrm{d}\tau$ and $\delta^{(3)}(y - z) = \delta(y^0 - z^0)\delta(y^1 - z^1)\delta(y^2 - z^2)$ is a three-dimensional Dirac distribution supported on the particle's world line ζ. Both the strength tensor $F^{\alpha\beta}$ and the current density j^α are evaluated at a field point $y \in \mathbb{M}_3$.

We express the electromagnetic field in terms of a vector potential, $\hat{F} = \mathrm{d}\hat{A}$. We impose the Lorentz gauge $A^\alpha{}_{,\alpha} = 0$; then the Maxwell field equation (2.1) becomes

$$\Box A^\alpha = -2\pi j^\alpha. \tag{2.3}$$

In 2+1 theory, the retarded Green's function associated with the D'Alembert operator $\Box := \eta^{\alpha\beta}\partial_\alpha\partial_\beta$ has support not just on the future light cone of the emission point x, but extends inside the light cone as well [1, 2]:

$$G^{\rm ret}_{2+1}(y,x) = \frac{\theta(y^0 - x^0 - |\mathbf{y}-\mathbf{x}|)}{\sqrt{-(y-x)^2}}, \tag{2.4}$$

where $\theta(y^0 - x^0 - |\mathbf{y}-\mathbf{x}|)$ is a step function defined to be one if $y^0 - x^0 \geqslant |\mathbf{y}-\mathbf{x}|$, and defined to be zero otherwise.

Convolving the retarded Green's function (2.4) with charge–current density $j^\alpha(x)$, we construct the retarded Liénard–Wiechert potential $A^{\rm ret}_\mu(y)$ in three dimensions. It is generated by the point charge during its entire past history before the retarded time $\tau^{\rm ret}(y)$ associated with the field point y. Apart from non-local term

$$F^{(\theta)}_{\mu\nu} = -e\int_{-\infty}^{\tau^{\rm ret}(y)} {\rm d}\tau \frac{u_\mu K_\nu - u_\nu K_\mu}{[-(K\cdot K)]^{3/2}}, \tag{2.5}$$

the strength tensor $F^{\rm ret}_{\mu\nu} = \partial_\mu A^{\rm ret}_\nu - \partial_\nu A^{\rm ret}_\mu$ of the adjunct electromagnetic field contains also local term

$$F^{(\delta)}_{\mu\nu} = \lim_{\tau\to\tau^{\rm ret}} \frac{e}{\sqrt{-(K\cdot K)}} \frac{u_\mu K_\nu - u_\nu K_\mu}{-(K\cdot u)}, \tag{2.6}$$

which is due to the differentiation of θ-function involved in $A^{\rm ret}_\mu(y)$. (By $K^\mu = y^\mu - z^\mu(\tau)$, we denote the unique timelike (or null) vector pointing from the emission point $z(\tau) \in \zeta$ to a field point y.)

The terms separately diverge on the light cone. But the singularity, however, can be removed from the sum of $\hat{F}^{(\delta)}$ and $\hat{F}^{(\theta)}$. Using the identity

$$\frac{1}{[-(K\cdot K)]^{3/2}} = \frac{1}{-(K\cdot u)}\frac{\rm d}{{\rm d}\tau}\frac{1}{\sqrt{-(K\cdot K)}} \tag{2.7}$$

in equation (2.5) yields

$$\begin{aligned} F^{(\theta)}_{\mu\nu} = & -\frac{e}{\sqrt{-(K\cdot K)}} \frac{u_\mu K_\nu - u_\nu K_\mu}{-(K\cdot u)}\bigg|_{\tau\to-\infty}^{\tau\to\tau^{\rm ret}(y)} \\ & + e\int_{-\infty}^{\tau^{\rm ret}(y)} \frac{{\rm d}\tau}{\sqrt{-(K\cdot K)}}\left\{\frac{a_\mu K_\nu - a_\nu K_\mu}{-(K\cdot u)} + \frac{u_\mu K_\nu - u_\nu K_\mu}{[-(K\cdot u)]^2}[1 + (K\cdot a)]\right\} \end{aligned} \tag{2.8}$$

after integration by parts. Taking into account that $1/\sqrt{-(K\cdot K)}$ vanishes whenever $\tau \to -\infty$,[1] we finally obtain the expression

$$\hat{F}^{\rm ret}(y) = e\int_{-\infty}^{\tau^{\rm ret}(y)} \frac{{\rm d}\tau}{\sqrt{-(K\cdot K)}}\left\{\frac{a\wedge K}{r} + \frac{u\wedge K}{r^2}[1 + (K\cdot a)]\right\} \tag{2.9}$$

which is regular on the light cone. It diverges on the particle's trajectory only. Symbol $\wedge$ denotes the wedge product. In 2+1 electrodynamics, outgoing waves propagate not just at the speed of light, but all speeds smaller than or equal to it. The invariant quantity

$$r = -(K\cdot u) \tag{2.10}$$

is an affine parameter on the timelike (null) geodesic that links y to $z(\tau)$; it can be loosely interpreted as the time delay between y and $z(\tau)$ as measured by an observer moving with the

[1] We assume that average velocities are not large enough to initiate particle creation and annihilation, so that 'space contribution' $|\mathbf{K}|$ cannot match with an extremely large zeroth component K^0.

particle. Because the speed of light is set to unity, parameter $r(\tau^{\rm ret})$ is also the spatial distance between $z(\tau^{\rm ret})$ and y as measured in this momentarily comoving Lorentz frame.

In three dimensions, the advanced Green's function is nonzero in the past of the emission point x. The advanced strength tensor

$$\hat{F}^{\rm adv}(y) = e\int_{\tau^{\rm adv}(y)}^{+\infty} \frac{{\rm d}\tau}{\sqrt{-(K\cdot K)}}\left\{\frac{a\wedge K}{r} + \frac{u\wedge K}{r^2}\left[1+(K\cdot a)\right]\right\} \tag{2.11}$$

is generated by the point charge during its entire future history following the advanced time associated with y.

3. Bound and radiative parts of Noether quantities

Equation of motion of radiating charge will be derived in the following section via analysis of the total flows of (retarded) electromagnetic field energy-momentum and angular momentum across a hyperplane $\Sigma_t = \{y\in\mathbb{M}_3 : y^0 = t\}$. Noether quantities are given by surface integrals:

$$p^\nu_{\rm em}(t) = \int_{\Sigma_t} {\rm d}\sigma_0 T^{0\nu} \tag{3.1}$$

$$M^{\mu\nu}_{\rm em}(t) = \int_{\Sigma_t} {\rm d}\sigma_0 (y^\mu T^{0\nu} - y^\nu T^{0\mu}). \tag{3.2}$$

The electromagnetic field's stress–energy tensor $\hat{T}$ has the components

$$2\pi T^{\mu\nu} = F^{\mu\lambda}F^\nu{}_\lambda - 1/4\eta^{\mu\nu}F^{\kappa\lambda}F_{\kappa\lambda}, \tag{3.3}$$

where $\hat{F}$ is the non-local strength tensor (2.9).

The computation is not a trivial matter, since the Maxwell energy-momentum tensor density evaluated at field point $y\in\Sigma_t$ is non-local. In odd dimensions, the retarded field is generated by the portion of the world line ζ that corresponds to the particle's history *before* $t^{\rm ret}(y)$. Since the stress–energy tensor is quadratic in field strengths, we should *twice* integrate it over ζ. We integrate it also over two variables which parametrize Σ_t in order to calculate energy-momentum and angular momentum which flow across this plane. The integrand describes the combination of outgoing electromagnetic waves emitted at instants t_1 and t_2 *before* t. (The situation is pictured in figure 1.) We trace a series of stages in calculations in the appendix; a detailed description is published in [18]. To summarize briefly, the integration of the energy-momentum and angular momentum tensor densities over variables which parametrize Σ_t results twofold integrals, we obtain two-point and three-point functions defined on the world line only, which are twice integrated over ζ. (See equations (A.31)–(A.33) in the appendix where the integration of the energy-momentum tensor density in 2+1 dimensions is considered.)

All the three-point terms which contain the observation instant t belong to the (diverging) bound parts of Noether quantities. They are permanently 'attached' to the charge and are carried along with it. The radiation parts contain the two-point functions depending on particle's position and velocity referred to the instants t_1 and t_2 before t.

Having done reparametrization, we rewrite the two-point functions which arise in energy-momentum and angular momentum in a manifestly covariant fashion:

$$p^\mu_{12} = u_{1,\alpha}\frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q\cdot q)]^{3/2}}, \qquad m^{\mu\nu}_{12} = z_1^\mu p^\nu_{12} - z_1^\nu p^\mu_{12}. \tag{3.4}$$

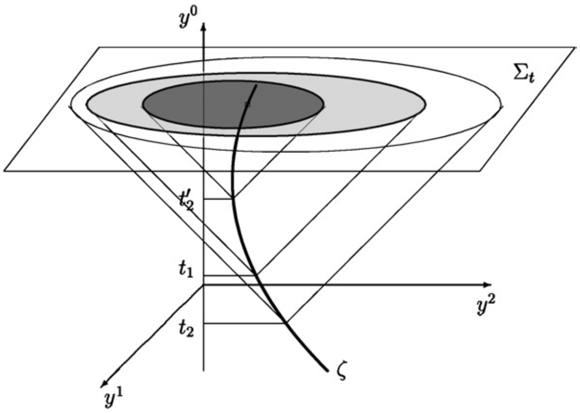

Figure 1. The integration of energy and momentum densities over the two-dimensional plane $y^0 = t$ means the study of interference of outgoing electromagnetic waves generated by different points on particle's world line. Let the instant t_1 be fixed. Outgoing electromagnetic waves generated by the portion of the world line that corresponds to the interval $-\infty < t_2 < t_1$ combine within the gray disc with radius $k_1^0 = t - t_1$. If the domain of integration $t_1 < t_2 \leqslant t$, the waves joint together inside the dark disc with radius $k_2^0 = t - t_2'$.

Index 1 indicates that the particle's velocity or position is referred to the instant $\tau_1 \in]-\infty, \tau]$ while index 2 says that the particle's characteristics are evaluated at instant $\tau_2 \leqslant \tau_1$. Here, $q^\mu = z_1^\mu - z_2^\mu$ defines the unique timelike geodesic connecting a field point $z(\tau_1) \in \zeta$ to an emission point $z(\tau_2) \in \zeta$. Since $q^\mu(\tau_2, \tau_1) = -q^\mu(\tau_1, \tau_2)$, the reciprocity relations are satisfied:

$$p_{21}^\mu\big|_{1\leftrightarrow 2} = -p_{12}^\mu, \qquad m_{21}^{\mu\nu}\big|_{1\leftrightarrow 2} = -m_{12}^{\mu\nu}. \tag{3.5}$$

The radiative parts of electromagnetic field's energy-momentum and angular momentum are as follows:

$$p_{\rm R}^\mu(\tau) = \frac{e^2}{2}\int_{-\infty}^{\tau} {\rm d}\tau_1 \int_{-\infty}^{\tau_1} {\rm d}\tau_2 \big(p_{12}^\mu + p_{21}^\mu\big), \tag{3.6}$$

$$M_{\rm R}^{\mu\nu}(\tau) = \frac{e^2}{2}\int_{-\infty}^{\tau} {\rm d}\tau_1 \int_{-\infty}^{\tau_1} {\rm d}\tau_2 \big(m_{12}^{\mu\nu} + m_{21}^{\mu\nu}\big). \tag{3.7}$$

Our next task will be to elucidate geometrical sense of these expressions.

We take the first terms under the integral signs in equations (3.6), (3.7) and denote them as $g_{12}^\alpha = \big(p_{12}^\mu, m_{12}^{\mu\nu}\big)$. We integrate the two-point functions (3.4) over the portion of the world line which corresponds to the interval $-\infty < \tau_2 \leqslant \tau_1$. We introduce the functions

$$G_{\rm ret}^\alpha(\tau_1) = e^2 \int_{-\infty}^{\tau_1} {\rm d}\tau_2\, g_{12}^\alpha. \tag{3.8}$$

Next we take the remaining terms g_{21}^α; we change the order of integration over the domain $D_\tau = \{(\tau_1, \tau_2) \in \mathbb{R}^2 : \tau_1 \in]-\infty, \tau], \tau_2 \leqslant \tau_1\}$:

$$\int_{-\infty}^{\tau} {\rm d}\tau_1 \int_{-\infty}^{\tau_1} {\rm d}\tau_2\, g_{21}^\alpha = \int_{-\infty}^{\tau} {\rm d}\tau_2 \int_{\tau_2}^{\tau} {\rm d}\tau_1\, g_{21}^\alpha. \tag{3.9}$$

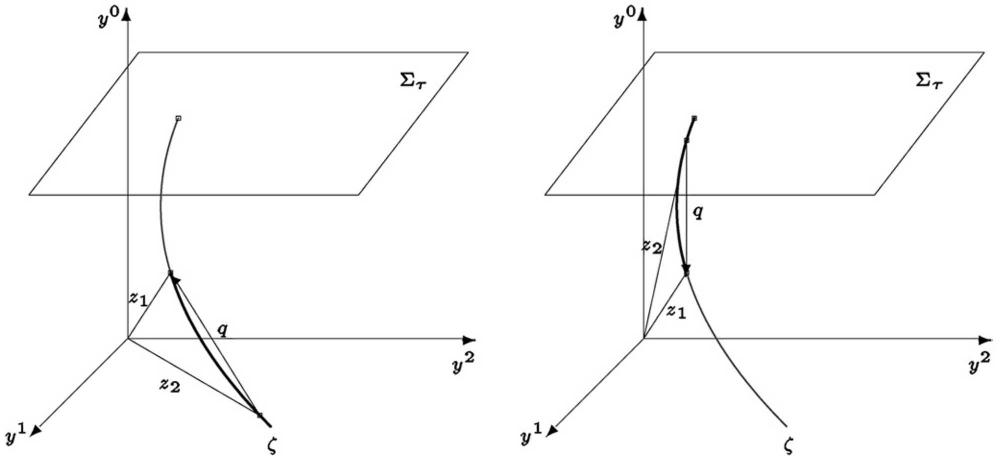

Figure 2. The term (3.8) with integration over the portion of the world line *before* τ_1 we call 'retarded'. The term (3.11) with integration over the portion of the world line *after* τ_1 we call 'advanced'. For an observer placed at point $z(\tau_1) \in \zeta$, the regular part (3.12) of Noether quantities carried by electromagnetic field looks as the combination of incoming and outgoing radiations. And yet the retarded causality is not violated. We still consider the interference of outgoing waves presented at the observation plane Σ_τ. The electromagnetic field carries information about the charge's past.

Since instants τ_1 and τ_2 label points at the same world line ζ, one can interchange indices 'first' and 'second' on the right-hand side of this equation. Taking into account relations (3.5), we finally obtain

$$\int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{-\infty}^{\tau_1} \mathrm{d}\tau_2\, g_{21}^{\alpha} = -\int_{-\infty}^{\tau} \mathrm{d}\tau_1 \int_{\tau_1}^{\tau} \mathrm{d}\tau_2\, g_{12}^{\alpha}. \tag{3.10}$$

The integrand coincides with that under integral sign on the right-hand side of equation (3.8) while the domain of inner integration corresponds to the interval $\tau_1 \leqslant \tau_2 \leqslant \tau$. We introduce the function

$$G_{\mathrm{adv}}^{\alpha}(\tau_1, \tau) = e^2 \int_{\tau_1}^{\tau} \mathrm{d}\tau_2\, g_{12}^{\alpha}. \tag{3.11}$$

We see that the double integral in equation (3.6) can be expressed as one-half of G_{ret} minus one-half of G_{adv} integrated over the world line ζ:

$$G_{\mathrm{R}}^{\alpha}(\tau) = \frac{1}{2} \int_{-\infty}^{\tau} \mathrm{d}\tau_1 \left[G_{\mathrm{ret}}^{\alpha}(\tau_1) - G_{\mathrm{adv}}^{\alpha}(\tau, \tau_1) \right]. \tag{3.12}$$

The situation is pictured in figure 2.

We evaluate the short-distance behaviour of the expression under the double integral in equation (3.12). Let τ_1 be fixed and $\tau_1 - \tau_2 := \Delta$ be a small parameter. With a degree of accuracy sufficient for our purposes

$$\sqrt{-(q \cdot q)} = \Delta, \qquad q^{\mu} = \Delta \left[u_1^{\mu} - a_1^{\mu} \frac{\Delta}{2} + \dot{a}_1^{\mu} \frac{\Delta^2}{6} \right], \qquad u_2^{\mu} = u_1^{\mu} - a_1^{\mu} \Delta + \dot{a}_1^{\mu} \frac{\Delta^2}{2}. \tag{3.13}$$

Substituting these into integrand of the double integral of equation (3.12) and passing to the limit $\Delta \to 0$ yields regular expression

$$\lim_{\tau_2\to\tau_1}\left[\frac{1}{2}u_{1,\alpha}\frac{-u_2^\alpha q^\mu + u_2^\mu q^\alpha}{[-(q\cdot q)]^{3/2}} + \frac{1}{2}u_{2,\alpha}\frac{-u_1^\alpha q^\mu + u_1^\mu q^\alpha}{[-(q\cdot q)]^{3/2}}\right] = \frac{1}{3}(a_1)^2 u_1^\mu - \frac{1}{12}\dot{a}_1^\mu. \tag{3.14}$$

Therefore, the subscript 'R' stands for 'regular' as well as 'radiative'.

Alternatively, choosing the linear superposition

$$G_{\rm S}^\alpha(\tau) = \frac{1}{2}\int_{-\infty}^{\tau}{\rm d}\tau_1\left[G_{\rm ret}^\alpha(\tau_1) + G_{\rm adv}^\alpha(\tau,\tau_1)\right], \tag{3.15}$$

we restore the singular parts of Noether quantities:

$$\begin{aligned} p_{\rm S}^\mu(\tau) &= \frac{e^2}{2}\int_{-\infty}^{\tau}{\rm d}\tau_2\frac{u^\mu(\tau_2)}{\sqrt{-(q\cdot q)}},\\ M_{\rm S}^{\mu\nu}(\tau) &= \frac{e^2}{2}\int_{-\infty}^{\tau}{\rm d}\tau_2\frac{z^\mu(\tau)u^\nu(\tau_2) - z^\nu(\tau)u^\mu(\tau_2)}{\sqrt{-(q\cdot q)}}. \end{aligned} \tag{3.16}$$

Since this non-local term diverges, the subscript 'S' stands for 'singular' as well as 'symmetric'.

4. Equation of motion of radiating charge

We therefore introduce the radiative part $p_{\rm R}$ of energy-momentum (see equation (3.6)) and postulate that it, and it alone, exerts a force on the particle. Singular part, $p_{\rm S}$, should be coupled with particle's three-momentum, so that 'dressed' charged particle would not undergo any additional radiation reaction. Already renormalized particle's individual three-momentum, say $p_{\rm part}$, together with $p_{\rm R}$ constitute the total energy-momentum of our composite particle plus field system: $P = p_{\rm part} + p_{\rm R}$. Since P does not change with time, its time derivative yields

$$\begin{aligned} \dot{p}_{\rm part}^\mu(\tau) &= -\dot{p}_{\rm R}^\mu\\ &= -\frac{e^2}{2}\int_{-\infty}^{\tau}{\rm d}s\left[u_{\tau,\alpha}\frac{-u_s^\alpha q^\mu + u_s^\mu q^\alpha}{[-(q\cdot q)]^{3/2}} + u_{s,\alpha}\frac{-u_\tau^\alpha q^\mu + u_\tau^\mu q^\alpha}{[-(q\cdot q)]^{3/2}}\right]. \end{aligned} \tag{4.1}$$

(The overdot means the derivation with respect to proper time τ.) Here, index τ indicates that the particle's velocity or position is referred to the observation instant τ while index s says that the particle's characteristics are evaluated at instant $s \leqslant \tau$.

Our next task is to derive expression which explain how the three-momentum of 'dressed' charged particle depends on its individual characteristics (velocity, position, mass, etc). We do not make any assumption about the particle structure, its charge distribution and its size. We only assume that the particle three-momentum $p_{\rm part}$ is finite. To find out the desired expression, we analyse conserved quantities corresponding to the invariance of the theory under proper homogeneous Lorentz transformations. The total angular momentum, say M, consists of particle's angular momentum $z \wedge p_{\rm part}$ and radiative part (3.7) of angular momentum carried by electromagnetic field:

$$M^{\mu\nu} = z_\tau^\mu p_{\rm part}^\nu(\tau) - z_\tau^\nu p_{\rm part}^\mu(\tau) + M_{\rm R}^{\mu\nu}(\tau). \tag{4.2}$$

Having differentiated (4.2) and inserting equation (4.1), we arrive at the equality

$$u_\tau \wedge p_{\rm part} = \frac{e^2}{2}\int_{-\infty}^{\tau}{\rm d}s\frac{u_\tau \wedge u_s}{\sqrt{-(q\cdot q)}}. \tag{4.3}$$

Apart from usual velocity term, the simplest solution contains also singular contribution (3.16) from particle's electromagnetic field:

$$\begin{aligned} p^\mu_{\text{part}}(\tau) &= m_0 u^\mu(\tau) + \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \frac{u^\mu(s)}{\sqrt{-(q\cdot q)}} \\ &= m_0 u^\mu(\tau) + p^\mu_S(\tau). \end{aligned} \tag{4.4}$$

Near the coincidence limit $s \to \tau$ the denominator of the integrand in equation (4.4) behaves as $\tau - s$, so that integral diverges logarithmically. Since p_{part} is proclaimed to be finite, the 'bare' mass m_0 should absorb divergency within the renormalization procedure. To cancel the infinity, Kazinski, Lyakhovich and Sharapov [2] add the term which, in our notations, looks as follows:

$$\delta m = \frac{e^2}{2}\int_{-\infty}^{\tau} \frac{\mathrm{d}s}{\sqrt{-(q\cdot q)}} \tag{4.5}$$

(see [2], equation (38)). Taking it into account we obtain

$$p^\mu_{\text{part}}(\tau) = m u^\mu(\tau) + \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \frac{u^\mu(s) - u^\mu(\tau)}{\sqrt{-(q\cdot q)}} \tag{4.6}$$

where already renormalized mass m is proclaimed to be finite.

The scalar product of particle three-velocity on the first-order time-derivative of particle three-momentum (4.1) is as follows:

$$(\dot p_{\text{part}}\cdot u_\tau) = \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \left[(u_\tau\cdot u_s)\frac{(u_\tau\cdot q)}{[-(q\cdot q)]^{3/2}} + \frac{(u_s\cdot q)}{[-(q\cdot q)]^{3/2}}\right]. \tag{4.7}$$

Since $(u\cdot a) = 0$, the scalar product of particle acceleration on the particle three-momentum (4.6) is given by

$$(p_{\text{part}}\cdot a_\tau) = \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \frac{(a_\tau\cdot u_s)}{\sqrt{-(q\cdot q)}}. \tag{4.8}$$

Summing up (4.7) and (4.8) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(p_{\text{part}}\cdot u_\tau) = \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \left\{\frac{\partial}{\partial\tau}\left[\frac{(u_\tau\cdot u_s)}{\sqrt{-(q\cdot q)}}\right] + \frac{(u_s\cdot q)}{[-(q\cdot q)]^{3/2}}\right\}. \tag{4.9}$$

Alternatively, the scalar product of three-momentum (4.6) and three-velocity is as follows:

$$(p_{\text{part}}\cdot u_\tau) = -m + \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \frac{(u_\tau\cdot u_s) + 1}{\sqrt{-(q\cdot q)}}. \tag{4.10}$$

Further we compare its differential consequence with equation (4.9). A surprising feature of the already renormalized *dynamical* mass m is that it depends on τ:

$$\dot m = \frac{e^2}{2}\int_{-\infty}^{\tau} \mathrm{d}s \frac{(q\cdot u_\tau) - (q\cdot u_s)}{[-(q\cdot q)]^{3/2}}. \tag{4.11}$$

It is interesting that similar phenomenon occurs in the theory which describes a point-like charge coupled with massless scalar field in flat spacetime of three dimensions [19]. The charge loses its mass through the emission of monopole radiation.

Having integrated derivative (4.11) over the world line ζ we obtain

$$\begin{aligned} m &= m_0 + \frac{e^2}{2}\int_{-\infty}^{\tau}\mathrm{d}\tau_1 \int_{-\infty}^{\tau_1}\mathrm{d}\tau_2 \left[\frac{\partial}{\partial\tau_1}\left(\frac{1}{\sqrt{-(q\cdot q)}}\right) + \frac{\partial}{\partial\tau_2}\left(\frac{1}{\sqrt{-(q\cdot q)}}\right)\right] \\ &= m_0 + \frac{e^2}{2}\int_{-\infty}^{\tau} \frac{\mathrm{d}s}{\sqrt{-(q\cdot q)}}, \end{aligned} \tag{4.12}$$

where m_0 is an infinite 'bare' mass of the particle. Inserting this into (4.6) we arrive at the equality $p^\mu_{\rm part}(\tau) = m_0 u^\mu_\tau + p^\mu_S$ which shows that particle's momentum renormalization agrees with the renormalization of mass.

The main goal of the present paper is to compute the effective equation of motion of radiating charge in 2+1 dimensions. To do it we replace $\dot p^\mu_{\rm part}$ on the left-hand side of equation (4.1) by differential consequence of equation (4.6) where the right-hand side of equation (4.11) substitutes for $\dot m$. At the end of a straightforward calculation, we obtain

$$ma^\mu_\tau = \frac{e^2}{2}a^\mu_\tau - e^2 u_{\tau,\alpha}\int_{-\infty}^{\tau} {\rm d}s \frac{-u^\alpha_s q^\mu + u^\mu_s q^\alpha}{[-(q\cdot q)]^{3/2}} + \frac{e^2}{2}a^\mu_\tau \int_{-\infty}^{\tau}\frac{{\rm d}s}{\sqrt{-(q\cdot q)}}. \tag{4.13}$$

The first term on the right-hand side of this equation looks horribly irrelevant. It arises also in ([2], equation (40)) where it is called 'the local part of Lorentz–Dirac (LD) force'. It is worth noting that Kazinski, Lyakhovich and Sharapov regularize the local part (2.6) of the retarded electromagnetic field and its non-local part (2.5) separately. (More exactly, the authors manipulate with convolution of these terms with particle's velocity taken at the instant τ.) Since the local part is proportional to particle's acceleration, they move it to the left-hand side of equation (2.1) where one-half of the squared charge is proclaimed to be absorbed by mass. The remaining non-local part (the second term in equation (2.1)) together with the third term of this equation constitute 'the renormalized LD force' which alone determines the radiation reaction.

But the finite first term in equation (2.1) is the integral part of Lorentz self-force and, therefore, cannot be cancelled within the regularization procedure. According to section 2, the sum of local (2.6) and non-local (2.8) parts of electromagnetic field results the non-local expression (2.9) for the retarded electromagnetic field. The field strengths at point $z(\tau)\in\zeta$ generated by the portion of the world line *before* the observation instant τ looks as follows:

$$\begin{aligned} F^{\mu\alpha}_{\rm ret}(\tau) &= e\int_{-\infty}^{\tau}\frac{{\rm d}s}{\sqrt{-(q\cdot q)}}\left\{\frac{u^\mu_s q^\alpha - u^\alpha_s q^\mu}{r^2}[1+(a_s\cdot q)] + \frac{a^\mu_s q^\alpha - a^\alpha_s q^\mu}{r}\right\} \\ &= \int_{-\infty}^{\tau}{\rm d}s\, f^{\mu\alpha}(\tau,s). \end{aligned} \tag{4.14}$$

Its convolution with particle's velocity is equal to the combination of the first and second terms on the right-hand side of equation (2.1):

$$eu_{\tau,\alpha}F^{\mu\alpha}_{\rm ret}(\tau) = \frac{e^2}{2}a^\mu_\tau - e^2 u_{\tau,\alpha}\int_{-\infty}^{\tau}{\rm d}s\frac{-u^\alpha_s q^\mu + u^\mu_s q^\alpha}{[-(q\cdot q)]^{3/2}}. \tag{4.15}$$

(It may be checked via using the equality (2.7) and integration by parts.) This relation prompts that the retarded Lorentz self-force should be substituted for this combination. If an external electromagnetic field $\hat F_{\rm ext}$ is applied, the equation of motion of radiating charge in 2+1 theory becomes

$$ma^\mu_\tau = eu_{\tau,\alpha}F^{\mu\alpha}_{\rm ret}(\tau) + \frac{e^2}{2}a^\mu_\tau\int_{-\infty}^{\tau}\frac{{\rm d}s}{\sqrt{-(q\cdot q)}} + eu_{\tau,\alpha}F^{\mu\alpha}_{\rm ext}. \tag{4.16}$$

The non-local term in equation (4.16) which is proportional to particle's acceleration $a(\tau)$ arises also in [2]. It gives rise to the renormalization of mass and provides proper short-distance behaviour of the perturbations due to the particle's own field. If $s\to\tau$ the integrand tends to the three-dimensional analogue of the Abraham radiation reaction vector:

$$\lim_{s\to\tau}\left[eu_{\tau,\alpha}f^{\mu\alpha}(\tau,s) + \frac{e^2}{2}\frac{a^\mu_\tau}{\sqrt{-(q\cdot q)}}\right] = \frac{2}{3}e^2(\dot a^\mu - a^2u^\mu). \tag{4.17}$$

(All quantities on the right-hand side refer to the instant of observation τ.)

If one moves the second term to the left-hand side of equation (4.16), they restore unphysical motion equation which follows from variational principle: it involves an infinite 'bare' mass and divergent Lorentz self-force.

5. Conclusions

In the present paper, we calculate the total flows of (retarded) electromagnetic field energy, momentum and angular momentum which flow across the plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$. We integrate the stress–energy tensor over two variables which parametrize Σ_t. Thanks to integration we reduce the support of the retarded and the advanced Green's functions to particle's trajectory.

The Dirac scheme which manipulates fields *on the world line only* is the key point of investigation. By fields we mean the convolution $eu_\nu(\tau_1)F^{\nu\mu}_{(\theta)}$ of three-velocity and non-local part (2.5) of the retarded strength tensor evaluated at point $z(\tau_1) \in \zeta$; the torque of this 'Lorentz θ-force' arises in electromagnetic field's total angular momentum. (Singular δ-term (2.6) is defined on the light cone; it is meaningless since both the field point, $z(t_1)$, and the emission point, $z(t_2)$, lie on the timelike world line.) The retarded and advanced quantities arise naturally. The retarded Lorentz self-force as well as its torque contain integration over the portion of the world line which corresponds to the interval $-\infty < t_2 \leqslant t_1$. Domain of integration of their 'advanced' counterparts corresponds to the interval $t_1 \leqslant t_2 \leqslant t$.

Noether quantity $G^\alpha_{\rm em}$ carried by electromagnetic field consists of terms of two quite different types: (i) singular, $G^\alpha_{\rm S}$, which is permanently 'attached' to the source and carried along with it; (ii) radiative, $G^\alpha_{\rm R}$, which detaches itself from the charge and leads an independent existence. The former is the half-sum of the retarded and advanced expressions, integrated over ζ, while the latter is the integral of one-half of the retarded quantity minus one-half of the advanced one. Within the regularization procedure, the bound terms $G^\alpha_{\rm S}$ are coupled with the energy-momentum and angular momentum of 'bare' source, so that already renormalized characteristics $G^\alpha_{\rm part}$ of charged particle are proclaimed to be finite. Noether quantities which are properly conserved become

$$G^\alpha = G^\alpha_{\rm part} + G^\alpha_{\rm R}.$$

The energy-momentum balance equations define the change of particle's three-momentum under the influence of an external electromagnetic field where loss of energy due to radiation is taken into account. The angular momentum balance equations explain how this already renormalized three-momentum depend on particle's individual characteristics. They constitute the system of three linear equations in three components of particle's momentum. Its rank is equal to 2, so that an arbitrary scalar function arises naturally. It can be interpreted as a *dynamical* mass of 'dressed' charge which is proclaimed to be finite. A surprising feature is that this mass depends on the particle's history before the instant of observation when the charge is accelerated. Already renormalized particle's momentum contains, apart from usual velocity term, also non-local contribution from point-like particle's electromagnetic field.

Having substituted this expression in the energy-momentum balance equations, we derive the three-dimensional analogue of the Lorentz–Dirac equation

$$ma^\mu_\tau = eu_{\tau,\alpha}F^{\mu\alpha}_{\rm ret}(\tau) + \frac{e^2}{2}a^\mu_\tau\int_{-\infty}^{\tau}\frac{{\rm d}s}{\sqrt{-(q\cdot q)}} + eu_{\tau,\alpha}F^{\mu\alpha}_{\rm ext}.$$

The loss of energy due to radiation is determined by the work done by the Lorentz force of point-like charge acting upon itself. Non-local term which is proportional to particle's acceleration provides finiteness of the self-action. It is intimately connected with the renormalization of mass. The third term describes influence of an external field.

In [1, 2], the Lorentz self-force is replaced with the second term of the right-hand side of equation (4.15) which does not possess quite clear physical sense. Besides, in [1] the non-local term which gives rise to the infinite mass renormalization and provides proper short-distance behaviour is as follows:

$$\frac{e^2}{2} a_\tau^\mu \int_{-\infty}^{\tau} \frac{\mathrm{d}s}{|\tau - s|}.$$

As noted in [2], there is a little sense in finiteness near the coincidence limit $s \to \tau$ since the expression for radiation reaction is not invariant with respect to reparametrization.

In this paper, we develop a convenient technique which allows us to integrate the non-local stress–energy tensor over the spacelike plane. The next step will be to implement this strategy to a point particle coupled to massive scalar field following an arbitrary trajectory on a flat spacetime. The Klein–Gordon field generated by the scalar charge holds energy near the particle. This circumstance makes unclear the procedure of decomposition of the energy-momentum into bound and radiative parts.

In [20] the remarkable correspondence is established between dynamical equations which govern the behaviour of superfluid ^{4}He films and Maxwell equations for electrodynamics in 2+1 dimensions (see also [21, 22])[2]. Perhaps the effective equation of motion (4.16) will be useful in the study of phenomena in superfluid dynamics which correspond to the radiation friction in 2+1 electrodynamics.

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Appendix. Energy-momentum of electromagnetic field in 2+1 dimensions

In this section, we trace a series of stages in calculation of the surface integral

$$p_{\mathrm{em}}^{\nu}(t) = \int_{\Sigma_t} \mathrm{d}\sigma_0 T^{0\nu} \tag{A.1}$$

which gives the energy-momentum carried by electromagnetic field of a point-like source arbitrarily moving in $\mathbb{M}_3$. Calculation of total angular momentum is virtually identical to that presented below, and we shall not bother with the details.

The Huygens principle does not hold in three dimensions: a point $z(t_1) \in \zeta$ produces the disc of radius $t - t_1$ in the observation plane $\Sigma_t = \{y \in \mathbb{M}_3 : y^0 = t\}$ (see figure 1). The integration of energy and momentum densities over Σ_t means the study of interference of outgoing electromagnetic waves emitted by different points on ζ:

$$p_{\mathrm{em}}^{\alpha} = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int_0^{k_1^0} \mathrm{d}R \int_0^{2\pi} \mathrm{d}\varphi J t_{12}^{0\alpha} + \int_{-\infty}^{t} \mathrm{d}t_1 \int_{t_1}^{t} \mathrm{d}t_2 \int_0^{k_2^0} \mathrm{d}R \int_0^{2\pi} \mathrm{d}\varphi J t_{12}^{0\alpha}. \tag{A.2}$$

The first multiple integral calculates the interference of the disc emanated by fixed point $z(t_1) \in \zeta$ with radiation generated by all the points *before* the instant t_1. The second fourfold integral gives the contribution of points *after* t_1.

[2] I wish to thank O Derzhko for drawing these papers to my attention.

The integrand

$$2\pi t_{12}^{\alpha\beta} = f_{(1)}^{\alpha\lambda} f_{(2)\lambda}^{\beta} - \tfrac{1}{4}\eta^{\alpha\beta} f_{(1)}^{\mu\nu} f_{\mu\nu}^{(2)} \tag{A.3}$$

describes the combination of field strength densities at $y \in \Sigma_t$

$$\hat{f}_{(a)}(y) = \frac{e}{\sqrt{-(K_a\cdot K_a)}}\left(\frac{\dot{v}_a \wedge K_a}{r_a} + \frac{v_a \wedge K_a}{(r_a)^2}\left[\gamma_a^{-2} + (K_a\cdot\dot{v}_a)\right]\right) \tag{A.4}$$

generated by emission points $z(t_1) \in \zeta$ and $z(t_2) \in \zeta$. The Jacobian J corresponds to the coordinate transformation

$$y^0 = t \qquad y^i = \alpha z^i(t_1) + \beta z^i(t_2) + R\omega^i{}_j n^j, \tag{A.5}$$

where $\alpha + \beta = 1$ and $n^j = (\cos\varphi, \sin\varphi)$. Orthogonal matrix

$$\omega = \begin{pmatrix} n_q^1 & -n_q^2 \\ n_q^2 & n_q^1 \end{pmatrix} \tag{A.6}$$

rotates space axes till new y^1-axis be directed along two-vector $\mathbf{q} := \mathbf{z}(t_1) - \mathbf{z}(t_2)$. (We denote $n_q^i = q^i/q$.)

It is worth noting that time variables t_1 and t_2 parametrize the same world line ζ. The coordinate transformation (A.5) is invariant with respect to the following reciprocity:

$$\Upsilon : t_1 \leftrightarrow t_2, \qquad \alpha \leftrightarrow \beta, \qquad \varphi \mapsto \varphi + \pi. \tag{A.7}$$

This symmetry provides identity of domains of fourfold integrals in energy-momentum (A.2).

It is obvious that the support of double integral $\int_{-\infty}^{t} \mathrm{d}t_1 \int_{t_1}^{t} \mathrm{d}t_2$ coincides with the support of the integral $\int_{-\infty}^{t} \mathrm{d}t_2 \int_{-\infty}^{t_2} \mathrm{d}t_1$. Since instants t_1 and t_2 label different points at the same world line ζ, one can interchange the indices 'first' and 'second' in the second fourfold integral of equation (A.2). Via interchanging these indices, we finally obtain $\int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2$ instead of initial $\int_{-\infty}^{t} \mathrm{d}t_1 \int_{t_1}^{t} \mathrm{d}t_2$. Taking into account these circumstances in the expression (A.2) for energy-momentum carried by electromagnetic field, we finally obtain

$$p_{\mathrm{em}}^{\alpha} = \int_{-\infty}^{t} \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \int_0^{k_1^0} \mathrm{d}R \int_0^{2\pi} \mathrm{d}\varphi\, J t^{0\alpha} \tag{A.8}$$

where the new stress–energy tensor is symmetric in the pair of indices 1 and 2:

$$\begin{aligned} 2\pi t^{0\alpha} &= 2\pi\left(t_{12}^{0\alpha} + t_{21}^{0\alpha}\right) \\ &= f_{(1)}^{0\lambda} f_{(2)\lambda}^{\alpha} + f_{(2)}^{0\lambda} f_{(1)\lambda}^{\alpha} - \tfrac{1}{4}\eta^{0\alpha}\left[f_{(1)}^{\mu\nu} f_{\mu\nu}^{(2)} + f_{(2)}^{\mu\nu} f_{\mu\nu}^{(1)}\right]. \end{aligned} \tag{A.9}$$

We see that it is sufficient to consider the situation when $t_1 \geqslant t_2$ pictured in figure A1. The gray disc with radius $k_1^0 = t - t_1$ is filled up by non-concentric circles with radii $R \in \left[0, k_1^0\right]$. Points in an R-circle are distinguished by angle φ.

To calculate the total flows (A.8) of the electromagnetic field energy and momentum which flow across the plane Σ_t, we should integrate the Maxwell energy-momentum tensor density (A.9) over angular variable φ, over radius R and, finally, over time variables t_1 and t_2. Integration over φ is a difficult undertaking. The difficulty resides mostly with norms $\|K_a\|^2 = -\eta_{\alpha\beta} K_a^\alpha K_a^\beta$ of separation vectors $K_a = y - z_a$ which result in elliptic integrals. To avoid dealing with them we modify the coordinate transformation (A.5). We fix the parameter β in such a way that the norm $\|K_1\|^2$ becomes proportional to the norm $\|K_2\|^2$:

$$\|K_1\|^2 = -\frac{\beta}{\alpha}\|K_2\|^2. \tag{A.10}$$

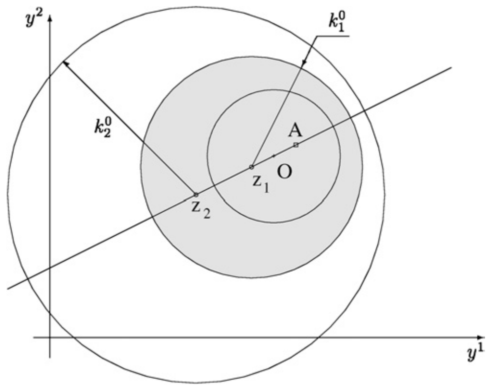

Figure A1. The interference picture in the plane Σ_t. The points $z(t_1) \in \zeta$ and $z(t_2) \in \zeta$, $t_2 < t_1$, emanate the radiation which filled up the discs centred at z_1 and z_2, respectively. The gray disc with radius $k_1^0 = t - t_1$ is filled up by non-concentric circles centred at the line crossing both the points z_1 and z_2. If parameter β vanishes the circle is centred at z_1, its radius is equal to k_1^0. If $\beta = \beta_0 < 0$, the circle reduces to the point A labelled by the box symbol. In case of intermediate value $\beta_0 < \beta < 0$, we have the circle of radius R with the centre at point O between z_1 and A.

Keeping in mind the identity $\alpha + \beta = 1$, we arrive at the quadratic algebraic equation on β which does not contain the angle variable:

$$R^2 = \alpha\left(k_1^0\right)^2 + \beta\left(k_2^0\right)^2 - \alpha\beta\mathbf{q}^2. \tag{A.11}$$

We choose the root which vanishes when $R = k_1^0$:

$$\begin{aligned} \beta &= \frac{1}{2\mathbf{q}^2}\left(-\left(k_2^0\right)^2 + \left(k_1^0\right)^2 + \mathbf{q}^2 + \sqrt{D}\right), \\ D &= \left[\left(k_2^0\right)^2 - \left(k_1^0\right)^2 - \mathbf{q}^2\right]^2 - 4\mathbf{q}^2\left[\left(k_1^0\right)^2 - R^2\right]. \end{aligned} \tag{A.12}$$

If $\mathbf{q}^2$ tends to zero while $t_1 \neq t_2$, it becomes the unique root of the linear equation on β originated from equation (A.11) with $\mathbf{q}^2 = 0$.

If $R = 0$, the R-circle reduces to the point A with the coordinates $\left(z_1^i - \beta_0 q^i\right)$, where $\beta_0 = \beta|_{R=0}$. If $R = k_1^0$ then $\beta = 0$ and the circle is centred at $z^i(t_1)$ (see figure A1).

Differential chart of the coordinate transformation (A.5) where $R(\beta)$ is given (implicitly) by equation (A.12) yields the Jacobian

$$J = (1/2)\left[\left(k_2^0\right)^2 - \left(k_1^0\right)^2 - \mathbf{q}^2\right] + \beta\mathbf{q}^2 - qR\cos\varphi. \tag{A.13}$$

To calculate the energy-momentum (A.8) carried by electromagnetic field, we should first perform the integration over the angle. When facing this problem it is convenient to mark out φ-dependent terms in expressions under the integral sign. In the Maxwell energy-momentum tensor density (A.9), we distinguish the second-order differential operator

$$\hat{\mathcal{T}}^a = \mathcal{D}^a\frac{\partial^2}{\partial t_1 \partial t_2} + \mathcal{B}^a\frac{\partial}{\partial t_1} + \mathcal{C}^a\frac{\partial}{\partial t_2} + \mathcal{A}^a \tag{A.14}$$

which has been labelled according to its dependence on the combination of components of the separation vectors K_1 and K_2, or on the Jacobian (A.13). The components of these vectors are involved in φ-dependent coefficients

$$\mathcal{D}^a = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{a}{r_1 r_2}, \qquad \mathcal{B}^a = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{ac_2}{r_1(r_2)^2},$$
$$\mathcal{C}^a = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{ac_1}{(r_1)^2 r_2}, \qquad \mathcal{A}^a = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\varphi \frac{ac_1c_2}{(r_1)^2(r_2)^2}, \tag{A.15}$$

where the factor a is replaced by $K_1^\alpha K_2^\beta, K_1^\alpha, K_2^\beta, J$ or 1 for $\hat{\mathcal{T}}_{12}^{\alpha\beta}, \hat{\mathcal{T}}_1^\alpha, \hat{\mathcal{T}}_2^\beta, \hat{\mathcal{T}}^J$ or $\hat{\mathcal{T}}^0$, respectively.

To distinguish the partial derivatives in time variables, we rewrite the operator (A.14) as the sum of the second-order differential operator

$$\hat{\Pi}^a = \frac{\partial^2}{\partial t_1 \partial t_2}\mathcal{D}^a + \frac{\partial}{\partial t_1}\left(\mathcal{B}^a - \frac{\partial \mathcal{D}^a}{\partial t_2}\right) + \frac{\partial}{\partial t_2}\left(\mathcal{C}^a - \frac{\partial \mathcal{D}^a}{\partial t_1}\right) \tag{A.16}$$

and the 'tail'

$$\pi^a = \frac{\partial^2 \mathcal{D}^a}{\partial t_1 \partial t_2} - \frac{\partial \mathcal{B}^a}{\partial t_1} - \frac{\partial \mathcal{C}^a}{\partial t_2} + \mathcal{A}^a. \tag{A.17}$$

For a smooth function $f(t_1, t_2)$, we have

$$\hat{\mathcal{T}}^a(f) = \hat{\Pi}^a(f) + f\pi^a. \tag{A.18}$$

Cumbersome calculations which is presented in [18] give the expressions

$$\pi^0 = 0, \qquad \pi^J = 0,$$
$$\pi_1^\alpha = v_1^\alpha\left(\mathcal{B}^0 - \frac{\partial \mathcal{D}^0}{\partial t_2}\right), \qquad \pi_2^\beta = v_2^\beta\left(\mathcal{C}^0 - \frac{\partial \mathcal{D}^0}{\partial t_1}\right),$$
$$\pi_1^{\alpha J} = v_1^\alpha\left(\mathcal{B}^J - \frac{\partial \mathcal{D}^J}{\partial t_2}\right), \qquad \pi_2^{\beta J} = v_2^\beta\left(\mathcal{C}^J - \frac{\partial \mathcal{D}^J}{\partial t_1}\right), \tag{A.19}$$
$$\pi_{12}^{\alpha\beta} = v_1^\alpha\left(\mathcal{B}_2^\beta - \frac{\partial \mathcal{D}_2^\beta}{\partial t_2}\right) + v_2^\beta\left(\mathcal{C}_1^\alpha - \frac{\partial \mathcal{D}_1^\alpha}{\partial t_1}\right) - v_1^\alpha v_2^\beta \mathcal{D}^0,$$

which allow us to rewrite the integral of $Jt^{\alpha\beta}$ over φ in terms of differential operators $\hat{\Pi}^a$. So, the integral of energy density t^{00} over the angular variable has the form

$$\int_0^{2\pi} \mathrm{d}\varphi J t^{00} = e^2[I\hat{\Pi}^0(\kappa) - I'\hat{\Pi}^J(\mu)], \tag{A.20}$$

where functions

$$\kappa = k_1^0 k_2^0 \frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + k_1^0 \frac{\partial \sigma}{\partial t_1} + k_2^0 \frac{\partial \sigma}{\partial t_2} - \frac{1}{2}\frac{\partial \sigma}{\partial t_1}\frac{\partial \sigma}{\partial t_2} + \sigma\mu,$$
$$\mu = \frac{1}{2}\frac{\partial^2 \sigma}{\partial t_1 \partial t_2} + 1. \tag{A.21}$$

World function $\sigma(t_1, t_2)$ of two timelike related points, $z(t_1) \in \zeta$ and $z(t_2) \in \zeta$, is equal to one-half of the square of vector $q^\mu = z^\mu(t_1) - z^\mu(t_2)$, taken with opposite sign:

$$\sigma(\tau_1, \tau_2) = -\tfrac{1}{2}(q \cdot q). \tag{A.22}$$

Symbols I, I' denote β-dependent factors

$$I = \frac{1}{\sqrt{-\beta\alpha}}, \qquad I' = \sqrt{\frac{-\beta}{\alpha}} + \sqrt{\frac{\alpha}{-\beta}}. \tag{A.23}$$

The mixed spacetime components of the stress–energy tensor (A.9) have the form

$$\begin{aligned}\int_0^{2\pi} \mathrm{d}\varphi J t^{0i} = \frac{e^2}{2} I \Bigg[\hat{\Pi}_1^i \left(\frac{\partial\lambda_2}{\partial t_1}\right) + \hat{\Pi}_2^i \left(\frac{\partial\lambda_1}{\partial t_2}\right) + \hat{\Pi}^0 \left(v_2^i \lambda_1 + v_1^i \lambda_2\right) \\ - \frac{\partial}{\partial t_1}\left(v_2^i \frac{\partial\lambda_1}{\partial t_2} \mathcal{D}^0\right) - \frac{\partial}{\partial t_2}\left(v_1^i \frac{\partial\lambda_2}{\partial t_1}\mathcal{D}^0\right)\Bigg] - \frac{e^2}{2} I' \hat{\Pi}^J \left(v_1^i + v_2^i\right),\end{aligned} \tag{A.24}$$

where

$$\lambda_1 = k_1^0 \frac{\partial\sigma}{\partial t_1} + \sigma, \qquad \lambda_2 = k_2^0 \frac{\partial\sigma}{\partial t_2} + \sigma. \tag{A.25}$$

We see that the integration of electromagnetic field's stress–energy tensor over φ yields integrals being functions of the end points only. In the following section, we classify them and consider the problem of integration over the remaining variables.

A.1. Integration over time variables and β

Our purpose in this section is to develop the mathematical tools required in a surface integration of the energy-momentum tensor density in 2+1 electrodynamics. Integration over angle variable results in the combination of partial derivatives in time variables:

$$p_{\mathrm{em}}^\alpha(t) = \left.\begin{matrix} e^2 \displaystyle\int_{-\infty}^t \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \\ e^2 \displaystyle\int_{-\infty}^t \mathrm{d}t_2 \int_{t_2}^{t} \mathrm{d}t_1 \end{matrix}\right\} \left(\frac{\partial^2 G_{12}^\alpha}{\partial t_1 \partial t_2} + \frac{\partial G_1^\alpha}{\partial t_1} + \frac{\partial G_2^\alpha}{\partial t_2}\right). \tag{A.26}$$

Two double integrals over (proper) time variables (one about the other) describe integration over the domain $D_t = \{(t_1, t_2) \in \mathbb{R}^2 : t_1 \in]-\infty, t], t_2 \leqslant t_1\}$.

By virtue of the equality

$$\int_{\beta_0}^0 \mathrm{d}\beta \frac{\partial G(\beta, t_1, t_2)}{\partial t_a} = \frac{\partial}{\partial t_a}\left[\int_{\beta_0}^0 \mathrm{d}\beta G(\beta, t_1, t_2)\right] + G(\beta_0, t_1, t_2)\frac{\partial\beta_0(t_1, t_2)}{\partial t_a}, \tag{A.27}$$

the triple integral (A.26) can be rewritten as follows:

$$\begin{aligned} p_{\mathrm{em}}^\alpha(t) = e^2 \left[\lim_{k_1^0 \to 0} \int_{\beta_0}^0 \mathrm{d}\beta G_{12}^\alpha\right]_{t_2 \to -\infty}^{t_2 = t} + e^2 \int_{-\infty}^t \mathrm{d}t_2 \lim_{k_1^0 \to 0}\left[G_{12}^\alpha\big|_{\beta=\beta_0} \frac{\partial\beta_0}{\partial t_2}\right] \\ - e^2 \int_{-\infty}^t \mathrm{d}t_2 \lim_{\triangle t \to 0} \int_{\beta_0}^0 \mathrm{d}\beta \left[\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha\right]_{k_1^0 = k_2^0 - \triangle t} + e^2 \int_{-\infty}^t \mathrm{d}t_2 \lim_{k_1^0 \to 0}\left[\int_{\beta_0}^0 \mathrm{d}\beta G_1^\alpha\right] \\ + e^2 \int_{-\infty}^t \mathrm{d}t_1 \lim_{\triangle t \to 0}\left[\int_{\beta_0}^0 \mathrm{d}\beta G_2^\alpha\right]_{k_2^0 = k_1^0 + \triangle t} - e^2 \int_{-\infty}^t \mathrm{d}t_1 \lim_{t_2 \to -\infty}\left[\int_{\beta_0}^0 \mathrm{d}\beta G_2^\alpha\right] \\ + \left.\begin{matrix} e^2 \displaystyle\int_{-\infty}^t \mathrm{d}t_1 \int_{-\infty}^{t_1} \mathrm{d}t_2 \\ e^2 \displaystyle\int_{-\infty}^t \mathrm{d}t_2 \int_{t_2}^{t} \mathrm{d}t_1 \end{matrix}\right\} \left(\left[\frac{\partial G_{12}^\alpha}{\partial t_2} + G_1^\alpha\right]_{\beta=\beta_0} \frac{\partial\beta_0}{\partial t_1} + G_2^\alpha\big|_{\beta=\beta_0} \frac{\partial\beta_0}{\partial t_2}\right). \end{aligned} \tag{A.28}$$

The functions under integral signs are as follows:

$$
\begin{aligned}
G^0_{12} &= I\mathcal{D}^0\kappa - I'\mathcal{D}^J\mu,\\
G^0_1 &= -\sqrt{\frac{-\beta}{\alpha}}\kappa\frac{\mathbf{v}_1^2}{\|r_1\|^3} - I'\mu\frac{\partial}{\partial\beta}\left(\frac{\beta}{\|r_1\|}\right),\\
G^0_2 &= \sqrt{\frac{\alpha}{-\beta}}\kappa\frac{\mathbf{v}_2^2}{\|r_2\|^3} - I'\mu\frac{\partial}{\partial\beta}\left(\frac{\alpha}{\|r_2\|}\right),
\end{aligned}
\tag{A.29}
$$

$$
\begin{aligned}
G^i_{12} &= \frac{I}{2}\left[\frac{\partial\lambda_1}{\partial t_2}\mathcal{D}^i_2 + \frac{\partial\lambda_2}{\partial t_1}\mathcal{D}^i_1 + \left(v^i_1\lambda_2 + v^i_2\lambda_1\right)\mathcal{D}^0\right] - \frac{I'}{2}\left(v^i_1 + v^i_2\right)\mathcal{D}^J,\\
G^i_1 &= \frac{I}{2}\frac{\beta}{\|r_1\|^3}\left[\frac{\partial\lambda_1}{\partial t_2}\left(\alpha q^i\mathbf{v}_1^2 + r^0_1 v^i_1\right) + \frac{\partial\lambda_2}{\partial t_1}\left(-\beta q^i\mathbf{v}_1^2 + r^0_1 v^i_1\right) + \left(v^i_1\lambda_2 + v^i_2\lambda_1\right)\mathbf{v}_1^2\right]\\
&\quad - \frac{I'}{2}\left(v^i_1 + v^i_2\right)\frac{\partial}{\partial\beta}\left(\frac{\beta}{\|r_1\|}\right),\\
G^i_2 &= \frac{I}{2}\frac{\alpha}{\|r_2\|^3}\left[\frac{\partial\lambda_1}{\partial t_2}\left(\alpha q^i\mathbf{v}_2^2 + r^0_2 v^i_2\right) + \frac{\partial\lambda_2}{\partial t_1}\left(-\beta q^i\mathbf{v}_2^2 + r^0_2 v^i_2\right) + \left(v^i_1\lambda_2 + v^i_2\lambda_1\right)\mathbf{v}_2^2\right]\\
&\quad - \frac{I'}{2}\left(v^i_1 + v^i_2\right)\frac{\partial}{\partial\beta}\left(\frac{\alpha}{\|r_2\|}\right).
\end{aligned}
\tag{A.30}
$$

All the terms involved in equation (A.28) possess specific small parameter. This circumstance allows us to expand the integrands into power series and perform the integration.

(i) *Integrals where $t_1 \to t$.* The lower limit β_0 tends to 0 if $k^0_1 = t - t_1$ vanishes. The upper limit is equal to zero too. Then the integral over parameter β vanishes whenever an expression under integral sign is smooth. So, we must limit our computations to the singular terms only. These integrals do not contribute in the energy-momentum at all.

(ii) *Integrals where $t_1 = t_2$.* The small parameter is the positively valued difference $\triangle t = t_1 - t_2$. The resulting terms belong to the bound part of energy-momentum. (To that which is permanently 'attached' to the charge and is carried along with it.)

(iii) *Integrals where $\mathbf{t}_2 \to -\infty$.* The lower limit β_0 tends to 0 if $k^0_2 = t - t_2$ increases extremely. Then the integral over parameter β vanishes whenever an expression under integral sign is smooth. So, we must limit our computations to the singular terms only. The resulting terms belong to the bound electromagnetic 'cloud' which cannot be separated from the charged particle.

(iv) *Integrals at point where $\beta = \beta_0$.* In this case, the radius of the smallest circle pictured in figure A1 vanishes and it reduces to the point A. The contribution in $p^\alpha_{\rm em}$ is given by the last line of equation (A.28). It can be rewritten as the combination of partial derivatives in time variables and non-derivative 'tail'. After integration over t_1 or $\mathbf{t}_2$, the derivatives are coupled with bound terms; the sum is absorbed by three-momentum of 'bare' particle within the renormalization procedure. The 'tail' contains radiative terms which detach themselves from the charge and lead an independent existence.

Summing up all the contributions (i)–(iv), we finally obtain

$$
\begin{aligned}
p^0_{\rm em}(t) &= e^2\frac{1 + 1/2\sqrt{1-\mathbf{v}_1^2}}{1+\sqrt{1-\mathbf{v}_1^2}}\frac{1}{\sqrt{1-\mathbf{v}_1^2}}\Bigg|_{t_1\to-\infty}^{t_1=t} + \frac{e^2}{2}\int_{-\infty}^{t}\mathrm{d}t_2\frac{1}{\sqrt{2\sigma(t,t_2)}} + e^2\int_{-\infty}^{t}\mathrm{d}t_1\int_{-\infty}^{t_1}\mathrm{d}t_2\\
&\quad\times\left[-\frac{(v_1\cdot v_2)q^0}{[2\sigma(t_1,t_2)]^{3/2}} + \frac{1}{2}\frac{(v_1\cdot q)}{[2\sigma(t_1,t_2)]^{3/2}} + \frac{1}{2}\frac{(v_2\cdot q)}{[2\sigma(t_1,t_2)]^{3/2}}\right],
\end{aligned}
\tag{A.31}
$$

$$p^i_{\rm em}(t) = e^2\,\frac{1+1/2\sqrt{1-\mathbf{v}_1^2}}{1+\sqrt{1-\mathbf{v}_1^2}}\,\frac{v_1^i}{\sqrt{1-\mathbf{v}_1^2}}\Bigg|_{t_1\to-\infty}^{t_1=t} + \frac{e^2}{2}\int_{-\infty}^{t}{\rm d}t_2\frac{v_2^i}{\sqrt{2\sigma(t,t_2)}} + e^2\int_{-\infty}^{t}{\rm d}t_1\int_{-\infty}^{t_1}{\rm d}t_2$$
$$\times\left[-\frac{(v_1\cdot v_2)q^i}{[2\sigma(t_1,t_2)]^{3/2}} + \frac{1}{2}\frac{(v_1\cdot q)v_2^i}{[2\sigma(t_1,t_2)]^{3/2}} + \frac{1}{2}\frac{(v_2\cdot q)v_1^i}{[2\sigma(t_1,t_2)]^{3/2}}\right], \tag{A.32}$$

where σ denotes the two-point function (A.22). The finite terms which depend on the end points only are non-covariant. They express the 'deformation' of electromagnetic 'cloud' due to the choice of the coordinate-dependent hole around the particle in the integration surface Σ_t. We neglect these structureless terms. The single integrals describe covariant singular part of energy-momentum carried by electromagnetic field. They arise from the following sum of 'three-point functions' which depend on particle's position and velocity referred to the instants t_1 and t_2 before the observation instant t as well as on t itself:

$$\frac{e^2}{2}\int_{-\infty}^{t}{\rm d}t_2\frac{v_2^\mu}{\sqrt{2\sigma(t,t_2)}} = \frac{e^2}{2}\int_{-\infty}^{t}{\rm d}t_2\,\frac{v_2^\mu}{\sqrt{(2t-t_1-t_2)^2-\mathbf{q}^2}}\Bigg|_{t_1=t_2}^{t_1=t}$$
$$+\frac{e^2}{2}\int_{-\infty}^{t}{\rm d}t_1\,\frac{v_1^\mu}{\sqrt{(2t-t_1-t_2)^2-\mathbf{q}^2}}\Bigg|_{t_2\to-\infty}^{t_2=t_1}. \tag{A.33}$$

(It is worth noting that the denominator evaluated at the remote past when $t_2 \to -\infty$ vanishes even if $t_1 \to -\infty$ too.) Two-point functions in between the square brackets of equations (A.31) and (A.32) determine the radiation reaction in 2+1 electrodynamics.

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